

- 2) A subset of $\mathbb{N}$ may look very different from a real number, but both can carry an infinite sequence of information. However, a subset records which natural numbers are included, while a real number can record digits.

This observation suggests to encode each subset as a distinct real number, then encode each real number as a distinct subset to prove:

$$|\mathcal{P}(\mathbb{N})| \geq |\mathbb{R}|$$

Let's begin by taking an arbitrary subset:

$$A = \{1, 3, 4, 7, \dots\}$$

Instead of listing its elements, record if each natural number belongs to it:

$$1, 0, 1, 1, 0, 0, 1, \dots$$

This binary sequence (where 1 means "belongs" and 0 means "does not belong") describes the subset, whether infinite or finite, completely.

We could try and turn this sequence into a binary number, but when dealing with sets tending to infinity:

$$(0.100\dots)_2 = (0.0111\dots)_2$$

Thus, completely different sequences could produce the same number. However, what makes this interesting is that we do not need to use every available digit.

We can use this sequence as the digits after the point in a base-three expansion. Our example therefore becomes:

$$F(\{1, 3, 4\}) = (0.101100\dots)_3 = \frac{1}{3} + \frac{1}{27} + \frac{1}{81}$$

For any subset A , this can be expressed by:

$$F(A) = \sum_{n=1}^{\infty} \frac{a_n}{3^n}, \quad a_n = \begin{cases} 1, & n \in A \\ 0, & n \notin A \end{cases}$$

We choose base 3 because, using only 0 and 1, a difference in any digit position cancels all possible difference in later positions to some extent:

$$\frac{1}{3^m} > \sum_{n=m+1}^{\infty} \frac{1}{3^n} = \frac{\frac{1}{3^{m+1}}}{1 - \frac{1}{3}} = \frac{1}{3^{m+1}} \cdot \frac{3}{2} = \frac{1}{2 \cdot 3^m}$$

This prevents two distinct subsets from receiving the same number. Moreover, every term is nonnegative, and the largest possible sum occurs when every digit is 1:

$$0 \leq F(A) \leq \sum_{n=1}^{\infty} \frac{1}{3^n} = \frac{\frac{1}{3}}{1 - \frac{1}{3}} = \frac{1}{2}$$

Thus, the construction assigns each subset a distinct real number in $[0, \frac{1}{2}]$.

Let B be a distinct subset of $\mathbb{N}$. ~~There~~ It follows that there must be a first index at which they differ. Call this position m and assume $m \in A$ but $m \notin B$. Assume:

$$1_A(m) = 1 \quad \text{and} \quad 1_B(m) = 0$$

All earlier digits agree, so upon subtraction, the difference at position m becomes $\frac{1}{3^m}$. Later digits could reduce this difference if B is 1 and A is 0.

However, even in the case of the greatest possible reduction:

$$F(A) - F(B) \geq \frac{1}{3^m} - \frac{1}{2 \cdot 3^m} = \frac{1}{2 \cdot 3^m} > 0$$

Thus, the later digits cannot cancel the first difference and $F(A) \neq F(B)$.

Thus, F is injective, proving $|P(\mathbb{N})| = |\mathbb{R}|$

For the reverse direction, we can use a different way of describing a real number. Let us consider its position relative to every rational number. For example, knowing a number is greater than 1 or 1.4 or 1.42 restricts its location due to identified bounds.

Since $\mathbb{Q}$ is countable, enumerate its elements without repetition:

$$q_1, q_2, q_3, \dots$$

For each $x \in \mathbb{R}$, define the subset

$$G(x) = \{n \in \mathbb{N} : q_n < x\}$$

This subset records which ^{rational} ~~natural~~ numbers lie below x . To see why two different real numbers give different subsets, suppose $x < y$. It follows that:

$$x < q_m < y$$

Here, m is the position of that rational number in the list. Since $q_m > x$, its index m is not included in $G(x)$. But since $q_m < y$, the same index is included in $G(y)$. The two subsets differ in at least one element, so

$$G(x) \neq G(y)$$

This works for every pair of distinct real numbers, however close they are. Therefore, G is injective as each real number is assigned a distinct subset of $\mathbb{N}$. Hence:

$$|\mathbb{R}| \leq |\mathcal{P}(\mathbb{N})|$$

If two sets inject into the other, it follows that a bijection exists between them. Thus,

$$|\mathcal{P}(\mathbb{N})| = |\mathbb{R}|$$

Hence proved.